Holographic Entanglement Entropy

Part III Seminar Talk

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- I may not be able to give satisfactory answers to some deep questions.

In this talk we will cover

1. Quantum Entanglement Entropy
2. Generalisation to Quantum Field Theory
3. Very Brief Introduction to AdS/CFT Correspondence
4. Ryu-Takayanagi Prescription
5. Applications in Black Hole Thermodynamics

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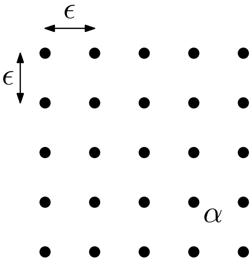
Quantum Lattice

Consider a lattice like this:



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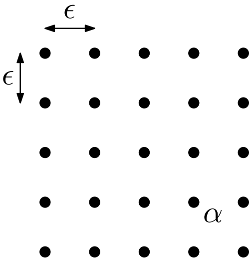

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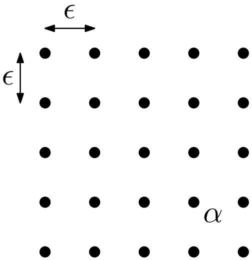

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$$\mathcal{H}_\alpha \simeq \mathcal{H}_{\text{qubit}} = \text{Span}\{|0\rangle, |1\rangle\}.$$

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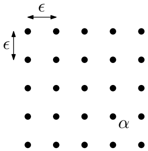

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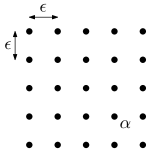

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i.e. as a linear combination of tensor products of the basis kets $|\psi_{\alpha,i_\alpha}\rangle$ on each lattice site α

$$|\Psi\rangle = \sum_{i_1,i_2,\cdots,i_\alpha,\cdots} C_{i_1 i_2 \cdots i_\alpha \cdots} |\psi_{1,i_1}\rangle \otimes |\psi_{2,i_2}\rangle \otimes \cdots \otimes |\psi_{\alpha,i_\alpha}\rangle \otimes \cdots .$$

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It has trace

$$\operatorname{tr}\rho := \sum_i \langle\phi_i|\rho|\phi_i\rangle = 1$$

where $\{|\phi_i\rangle\}$ form an orthonormal complete basis of $\mathcal{H}$.

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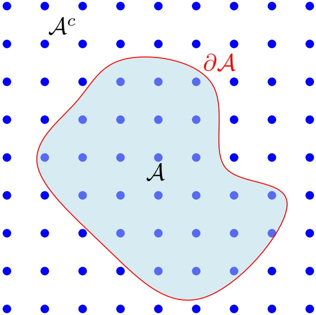

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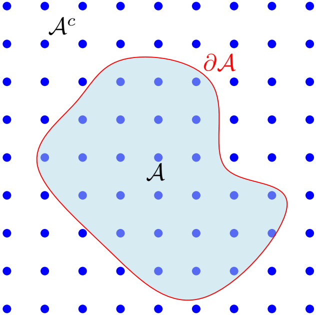

Now

$$\mathcal{H} \simeq \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{A}^c}.$$

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$$S_{\mathcal{A}}^{(q)} := \frac{1}{1-q} \log \left(\mathrm{tr}_{\mathcal{A}} \, \rho_{\mathcal{A}}^{q} \right) \quad \text{for} \quad q \in \mathbb{Z}_{+}.$$

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We analytically continue q to $\mathbb{R}_+$ (always assume it works!) so that

$$S_{\mathcal{A}} = \lim_{q \to 1} S_{\mathcal{A}}^{(q)}.$$

(Exercise: check this.)

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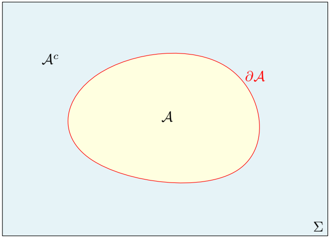

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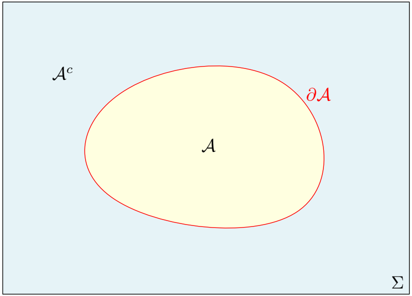

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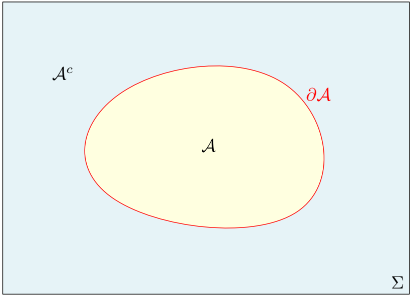

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Evolution of states in $\mathcal{B}_d$ fully determined by information on Σ_{d-1}.

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Time evolution is trivial, can set $t \to \mathrm{i}t$ so we have a Euclidean construction. WLOG, choose Σ_{d-1} as $t = 0$.

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Path Integral Formulation

We single out region $\mathcal{A}$ by specifying the field configurations $\Phi_{\mathcal{A}}(t=0) = \Phi_0$ and take the trace over $\mathcal{H}_{\mathcal{A}^c}$ by doing path integral

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In general, we want integer powers of $\rho_{\mathcal{A}}$. "Cut-up" the configuration of $\Phi_{\mathcal{A}}$ at $t = 0$ as

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so we can use *replica trick* later. Now the density matrix element is

$$(\rho_{\mathcal{A}})_{-+} = \int \mathcal{D}\Phi \; e^{-S_{\mathrm{QFT}}[\Phi]} \delta[\Phi_{\mathcal{A}}(t=0^-) - \Phi_-] \delta[\Phi_{\mathcal{A}}(t=0^+) - \Phi_+].$$

Replica Trick

Integer powers of the reduced matrix is obtained by taking q copies of $\mathcal{B}_d$ and match field configurations $\Phi_+^{(k)} = \Phi_-^{(k+1)}$ for $k = 1, 2, \cdots, (q-1)$.

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$$(\rho_{\mathcal{A}})^q_{-+} = \left(\int \prod_{j=1}^{q-1} \mathrm{d}\Phi_+^{(j)}\, \delta[\Phi_+^{(j)} - \Phi_-^{(j+1)}] \right)$$
$$\times \left(\int \prod_{k=1}^{q} \mathcal{D}\Phi^{(k)}\ e^{-S_{\mathrm{QFT}}[\Phi^{(k)}]} \delta[\Phi_{\mathcal{A}}^{(k)}(t=0^-) - \Phi_-^{(k)}] \delta[\Phi_{\mathcal{A}}^{(k)}(t=0^+) - \Phi_+^{(k)}] \right)$$

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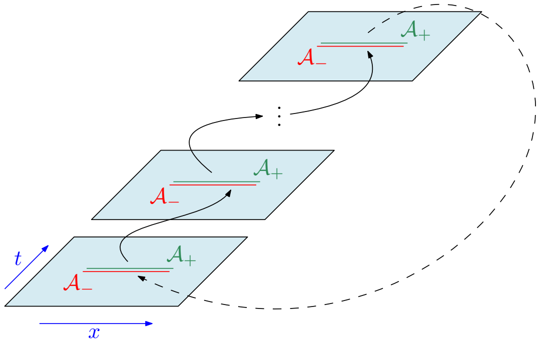

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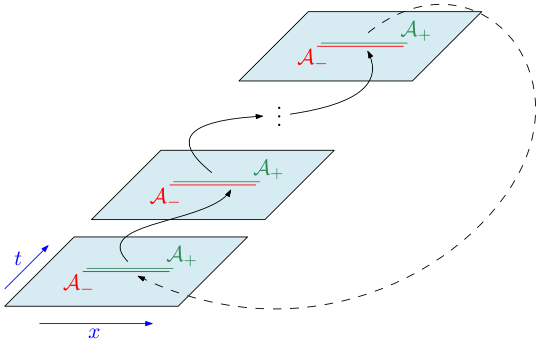

with the dashed line as taking trace of $\rho_{\mathcal{A}}^q$.

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where

$$\mathcal{Z}[\mathcal{M}] = \int_{\mathcal{M}} \mathcal{D}\Phi \ e^{-S_{\mathrm{QFT}}[\Phi]}$$

is the *partition function* on spacetime $\mathcal{M}$ and we chose $\mathcal{Z}[\mathcal{B}_d]^q$ as the normalisation factor for the trace of order q.

Entanglement Entropy in QFT

Hence, the Rényi entropy is

$$S_{\mathcal{A}}^{(q)} = \frac{1}{1-q}\left(\log \mathcal{Z}[\mathcal{B}_d^{(q)}] - q\log \mathcal{Z}[\mathcal{B}_d]\right).$$

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Analytically continue q to real numbers, we have the entanglement entropy in terms of path integrals as

$$S_{\mathcal{A}} = \lim_{q\to 1}\frac{1}{1-q}\left(\log \mathcal{Z}[\mathcal{B}_d^{(q)}] - q \log \mathcal{Z}[\mathcal{B}_d]\right).$$

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By *conformal* we mean the QFT is invariant under conformal transformation (Poincaré + dilation) with symmetry group $SO(2, d)$.

AdS/CFT Correspondence

Historically, such duality was originally discovered by Juan Maldacena[2] in String Theory, that

$$\mathcal{N} = 4, d = 4, \text{ SU}(N_c) \text{ super Yang-Mills}$$

$$\Updownarrow$$

$$\text{IIB supergravity with geometry AdS}_5 \times \mathbb{S}^5$$

If you know nothing (as me) about supersymmetry, it is okay. We don't need it here.

[2] Maldacena, *The Large-N Limit of Superconformal Field Theories and Supergravity*, Intl. Jrnl. Theo. Phys. **38** (1999), no. 1113-1133

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with $\mathcal{Y}$ some compact space which is required for a consistent embedding into string theory. It doesn't matter for our purpose today.

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No quantum, stringy effect, only classical gravity.

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"CFT_d lives on the boundary of AdS_{d+1}"

Bulk Dynamics

Bulk spacetime has action

$$S_{\text{bulk}} = \frac{1}{16\pi G_N^{(d+1)}} \int \mathrm{d}^{d+1}x \sqrt{-g} \left(R - 2\Lambda + \mathcal{L}_{\text{matter}} \right),$$

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leading to Einstein equations

$$G_{ab} + \Lambda g_{ab} = T_{ab}^{\text{matter}}$$

with boundary condition $\partial\mathcal{M}_{d+1} = \mathcal{B}_d$.

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Excited states are dual to non-trivial asymptotic AdS_{d+1}.
Metric by solving Einstein equation.

Partition Function

A powerful result to use later is

$$\mathcal{Z}_{\text{CFT}} = \mathcal{Z}_{\text{gravity}}.$$

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Now we know AdS/CFT correspondence. How to calculate entanglement entropy in $\mathcal{B}_d$ using it?

[3] Ryu and Takayanagi, *Holographic Derivation of Entanglement Entropy from the anti–de Sitter Space/Conformal Field Theory Correspondence*, Phys. Rev. Lett. **96** (2006) 181602

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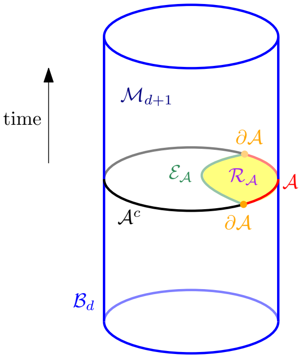
RT Proposal

For CFT$_d$ on $\mathcal{B}_d$, we want $S_\mathcal{A}$ of $\mathcal{A}$ on some Cauchy slice $\Sigma_{d-1} \subset \mathcal{B}_d$.

As before, we consider $\Sigma_{d-1} = \mathcal{A} \cup \mathcal{A}^c$ and call $\partial\mathcal{A}$ *entangling surface*.

RT Proposal

We need to find a surface $\mathcal{E}_\mathcal{A}$ which



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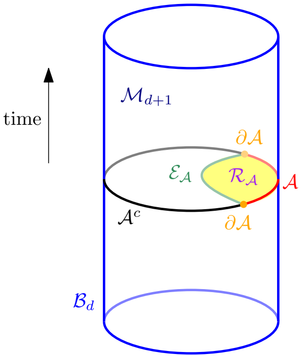

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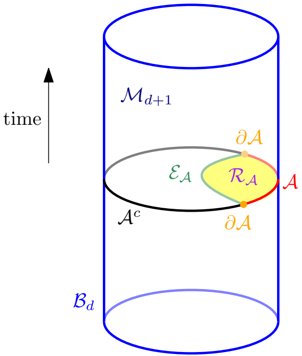

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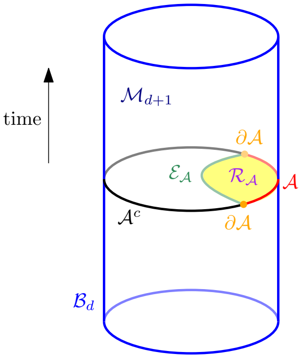

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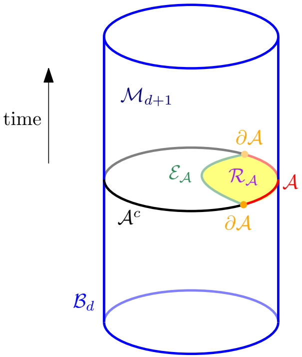
4. satisfies a *homology constraint*, i.e. $\mathcal{E}_{\mathcal{A}}$ is smoothly retractable to $\mathcal{A}$, or say, there exists a spacelike, bulk codimension-1, smooth interpolating surface $\mathcal{R}_{\mathcal{A}} \subset \mathcal{M}_{d+1}$ bounded by $\mathcal{E}_{\mathcal{A}}$ and $\mathcal{A}$.



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From all valid $\mathcal{E}_{\mathcal{A}}$, pick the one with smallest area.

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Then it can be shown

$$S_{\mathcal{A}} = \min_{\mathcal{E}_{\mathcal{A}}} \frac{\text{Area}(\mathcal{E}_{\mathcal{A}})}{4G_N^{(d+1)}}.$$

Generalisation of RT

The generalisations of RT prescription include

[5] Hubeny, Rangamani and Takayanagi, *A covariant holographic entanglement entropy proposal*, JHEP (2007)

[6] Wall, *Maximin surfaces, and the strong subadditivity of the covariant holographic entanglement entropy*, Classical and Quantum Gravity **31** (2014) 225007.

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We don't have time to elaborate on these. If interested, please refer to the original papers.

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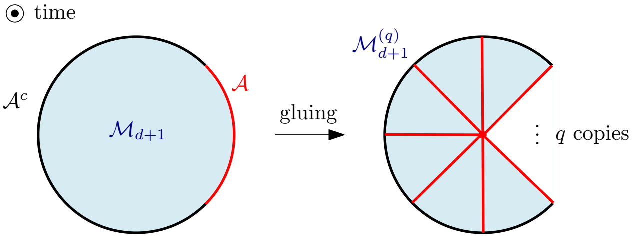

RT Derivation Sketch 2

We are in the classical limit in the bulk, so the partition function path integral can be approximated by saddle point, i.e.

$$\mathcal{Z}_{\text{gravity}}[\mathcal{M}_{d+1}^{(q)}] \approx e^{-I[\mathcal{M}_{d+1}^{(q)}]}$$

where $I[\mathcal{M}_{d+1}^{(q)}]$ means *on-shell* action.

RT Derivation Sketch 3

As the partition function $\mathcal{B}_d$ and $\mathcal{M}_{d+1}$ match, the Rényi entropy is given by the on-shell actions in the classical bulk gravity

$$S^{(q)} = \frac{1}{1-q}\left(\log \mathcal{Z}_{\text{gravity}}[\mathcal{M}_{d+1}^{(q)}] - q \log \mathcal{Z}_{\text{gravity}}[\mathcal{M}_{d+1}^{(1)}]\right)$$
$$\approx \frac{1}{q-1}\left(I[\mathcal{M}_{d+1}^{(q)}] - qI[\mathcal{M}_{d+1}^{(1)}]\right).$$

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Take the quotient $\hat{\mathcal{M}}^{(q)} = \mathcal{M}_{d+1}^{(q)}/\mathbb{Z}_q$, which is in fact an *orbifold*. It is topologically equivalent to $\mathcal{M}_{d+1}$.

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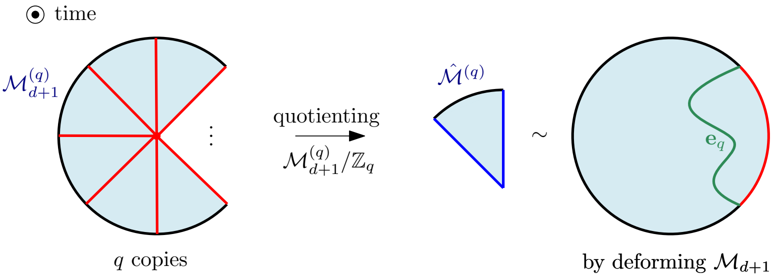

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$$S = \lim_{q\to 1} S^{(q)} = \left(q\partial_q I[\hat{\mathcal{M}}^{(q)}]\right)\Big|_{q=1}.$$

RT Derivation Sketch 5

It can be shown that

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We can use the replica trick to calculate the black hole entropy. Previous work by Xi Dong[7] has shown marvellous resolution of JKM ambiguity of BH entropy in higher curvature gravity.

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More to explore!

Thank You!