Part III Talk: **Holographic Entanglement Entropy**

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Abstract

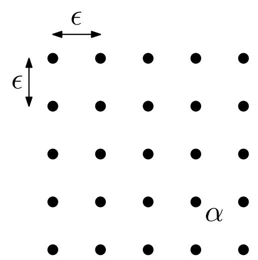
In this talk, we briefly introduce quantum entanglement entropy in a field theoretic perspective and review the holographic proposals for computing such entropy. We also concisely discuss its applications in calculating black hole entropy. This is a huge subject, we only introduce the central notions. Due to limited time, we will omit many details which can be referred to previous works listed in the references.

Disclaimer

Given the vastness and depth of this subject, this talk will only give a more qualitative introduction. Many technical details are omitted due to limited time and limited knowledge of the speaker. The speaker may not know enough to give satisfactory answers to some deep questions as he himself is still learning about this area.

1 Quantum Entanglement Entropy

To quickly show you the notion of *quantum entanglement entropy*, consider the following system: a lattice of points with separation ϵ.



On each site labelled by α, there is a finite-dimensional Hilbert space $\mathcal{H}_\alpha$. For example, we may take $\mathcal{H}_\alpha \simeq \mathcal{H}_{\text{qubit}}$, i.e. $\mathcal{H}_\alpha = \{|0\rangle, |1\rangle\}$. Now, the state of the whole system $|\Psi\rangle$ living in Hilbert space $\mathcal{H}$ satisfy

$$|\Psi\rangle \in \bigotimes_\alpha \mathcal{H}_\alpha,$$

i.e., it can be expressed as a linear combination of tensor products of the basis kets on each lattice site.

To further investigate entanglement, we first introduce the *density operator*. For a normalised state $|\Psi\rangle$, the density operator associated to it is defined as

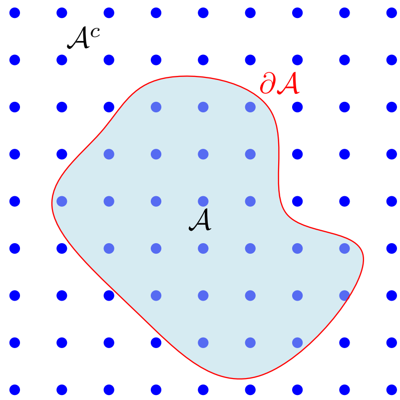
$$\rho = |\Psi\rangle\langle\Psi|.$$

It has trace

$$\operatorname{tr}\rho := \sum_i \langle\phi_i|\rho|\phi_i\rangle = 1.$$

where $\{|\phi_i\rangle\}$ form an orthonormal complete basis of $\mathcal{H}$.

The notion of *entanglement* arises when we consider the whole system in two separate pieces. We bipartition the lattice sites into two pieces, $\mathcal{A}$ and $\mathcal{A}^c$.



The Hilbert space itself can be seen as bipartitioned, i.e.

$$\mathcal{H} \simeq \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{A}^c}.$$

We define the *reduced density operator* with respect to region $\mathcal{A}$ as

$$\rho_{\mathcal{A}} := \operatorname{tr}_{\mathcal{A}^c}\rho$$

and further the *von Neumann entropy*, also known as *entanglement entropy*, is defined by

$$S_{\mathcal{A}} = -\operatorname{tr}_{\mathcal{A}}(\rho_{\mathcal{A}}\log\rho_{\mathcal{A}}).$$

By subscript $\mathcal{A}$ or $\mathcal{A}^c$ on the trace we are summing over only basis states of $\mathcal{H}_{\mathcal{A}}$ or $\mathcal{H}_{\mathcal{A}^c}$, correspondingly.

The intuition can be drawn from the Gibbs entropy in classical statistical mechanics and Shannon's information entropy, on which we will not elaborate.

The problem here is to rigorously define the logarithm of an operator.

To resolve such problem, we can take integer powers of such operator and see how to exploit them. This brings the definition of *Rényi entropy* (of order q)

$$S_{\mathcal{A}}^{(q)} := \frac{1}{1-q}\log\left(\operatorname{tr}_{\mathcal{A}}\rho_{\mathcal{A}}^q\right) \quad \text{for} \quad q \in \mathbb{Z}_+.$$

To validate the calculation of entanglement entropy, we would like to analytically continue q to $\mathbb{R}_+$. Later we will show the *replica trick* that does the work. Assume such analytic continuation is valid, we have

$$S_{\mathcal{A}} = \lim_{q\to 1} S_{\mathcal{A}}^{(q)}.$$

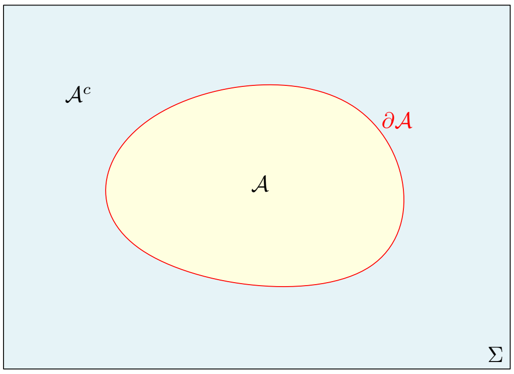
2 Generalisation to Quantum Field Theory

Now we would generalise the notion of entanglement entropy to quantum field theories. In a field theoretic perspective, we are working in a continuum of spacetime, namely we take the lattice spacing $\epsilon \to 0$. Also, we will not assume the Hilbert spaces at each site to be compact.

We work in a general d-dimensional Lorentzian spacetime $\mathcal{B}_d$ which is *globally hyperbolic*, i.e. there exists a *Cauchy surface*. To have a notion of *bipartitioning the spatial region*, we pick one Cauchy slide Σ_{d-1} which is spacelike, defining 'a moment of simultaneity'. Bipartition it such that

$$\Sigma_{d-1} = \mathcal{A} \cup \mathcal{A}^c$$

and denote the separatrix as $\partial\mathcal{A}$, which is a spacetime codimension-2 surface.



Knowing the field configuration $\Phi(x)$ on Σ_{d-1}, we can determine the evolution of the physics of whole spacetime $\mathcal{B}_d$, thus we have a state of the system on Σ_{d-1}, which can be seen as a wave functional $\Psi[\Phi(x)]$ for a pure state, or in general a density operator ρ_Σ.

Similar to the discrete case, we can decompose the Hilbert space of the QFT as $\mathcal{H} = \mathcal{H}_\mathcal{A} \otimes \mathcal{H}_{\mathcal{A}^c}$. However, it may involve subtleties such as gauge dependence, which we will not elaborate here. We want to generalise the notion of $\rho_\mathcal{A} = \mathrm{tr}_{\mathcal{H}_{\mathcal{A}^c}}(\rho_\Sigma)$ to discuss entanglement between region $\mathcal{A}$ and $\mathcal{A}^c$ in QFT.

To illustrate the basic idea, we only present the time-independent case. For more, please see Section 2.3 of [1].

For static states where the time evolution is trivial, we can use Euclidean construction (usually by $t \to it$). Wlog, choose the Cauchy slice as $t = 0$. To get the reduced density matrix, we are singling out region $\mathcal{A}$ by specifying the field configurations $\Phi_\mathcal{A}(t = 0) = \Phi_0$ and taking the trace over $\mathcal{H}_{\mathcal{A}^c}$ by doing a path integral, namely

$$\rho_\mathcal{A}[\Phi_0] = \int \mathcal{D}\Phi \; e^{-S_{\mathrm{QFT}}[\Phi]} \delta[\Phi_\mathcal{A}(t = 0) - \Phi_0].$$

However, we would like to consider integer powers of $\rho_\mathcal{A}$. It is more general to "cut-up" the configuration of $\Phi_\mathcal{A}$ at $t = 0$ as

$$\Phi_\mathcal{A}(t = 0^-) = \Phi_-, \quad \Phi_\mathcal{A}(t = 0^+) = \Phi_+$$

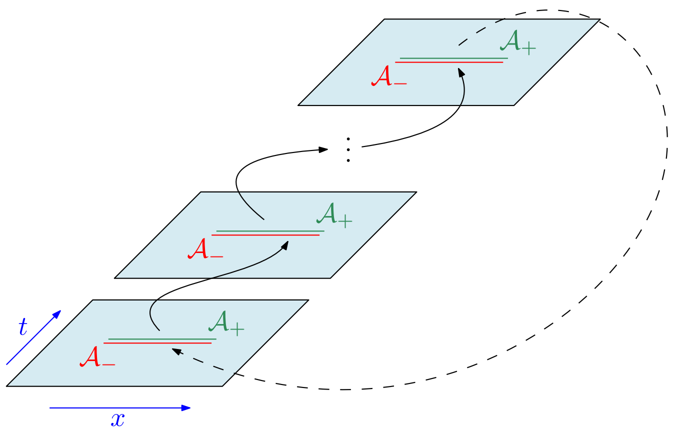
so that we can use the replica trick later. In this case, the density matrix element is

$$(\rho_\mathcal{A})_{-+} = \int \mathcal{D}\Phi \; e^{-S_{\mathrm{QFT}}[\Phi]} \delta[\Phi_\mathcal{A}(t = 0^-) - \Phi_-]\delta[\Phi_\mathcal{A}(t = 0^+) - \Phi_+].$$

To get powers of the reduced matrix, we take q copies of the spacetime $\mathcal{B}_d$ and match the field configurations by $\Phi_+^{(k)} = \Phi_-^{(k+1)}$ for $k = 1, 2, \cdots, (q-1)$. Thus we can write

$$(\rho_{\mathcal{A}})^q_{-+} = \left(\int \prod_{j=1}^{q-1} \mathrm{d}\Phi_+^{(j)}\, \delta[\Phi_+^{(j)} - \Phi_-^{(j+1)}] \right) \times \left(\int \prod_{k=1}^{q} \mathcal{D}\Phi^{(k)}\ e^{-S_{\mathrm{QFT}}[\Phi^{(k)}]} \delta[\Phi_{\mathcal{A}}^{(k)}(t=0^-) - \Phi_-^{(k)}] \delta[\Phi_{\mathcal{A}}^{(k)}(t=0^+) - \Phi_+^{(k)}] \right).$$

We can think this as gluing q copies of $\mathcal{B}_d$ together, as shown in the following figure.



We can see the resultant manifold as a whole, and call it $\mathcal{B}_d^{(q)}$. It is obvious that this manifold has a $\mathbb{Z}_q$ symmetry, also known as the *replica symmetry*.

To find the entanglement entropy, we need to evaluate the Rényi entropy first. Taking the trace of $\rho_{\mathcal{A}}^q$, this is equivalent to matching $\Phi_+^{(q)}$ with $\Phi_-^{(1)}$ and take all possible field configurations on different copies of $\mathcal{A}$'s. Then we can write

$$\mathrm{tr}\left(\rho_{\mathcal{A}}^q\right) = \frac{\mathcal{Z}[\mathcal{B}_d^{(q)}]}{\mathcal{Z}[\mathcal{B}_d]^q},$$

where

$$\mathcal{Z}[\mathcal{M}] = \int_{\mathcal{M}} \mathcal{D}\Phi\ e^{-S_{\mathrm{QFT}}[\Phi]}$$

is the partition function on spacetime $\mathcal{M}$ and we chose $\mathcal{Z}[\mathcal{B}_d]^q$ as the normalisation factor for the trace of order q.

Hence, the Rényi entropy is

$$S_{\mathcal{A}}^{(q)} = \frac{1}{1-q} \left(\log \mathcal{Z}[\mathcal{B}_d^{(q)}] - q \log \mathcal{Z}[\mathcal{B}_d] \right).$$

Here, we won't worry about the validity of analytic continuation of q to real numbers as the rigorous discussion is far beyond our scope of knowledge. Assuming it is valid, we have

$$S_{\mathcal{A}} = \lim_{q\to 1} \frac{1}{1-q} \left(\log \mathcal{Z}[\mathcal{B}_d^{(q)}] - q \log \mathcal{Z}[\mathcal{B}_d] \right).$$

3 Very Brief Introduction to AdS/CFT Correspondence

Although we've got a nice formula for the entanglement entropy in QFT, it is in general formidable to evaluate. Here we very briefly introduce the *AdS/CFT correspondence*, a.k.a. *holographic principle*, which provides alternative route to tackle the problem. The main argument of AdS/CFT correspondence is that the (quantum) gravity in $(d+1)$-dimensional anti de Sitter space AdS_{d+1} is *equivalent* to a d-dimensional conformal field theory CFT_d. This is duality was originally shown in string theory by Juan Maldacena [2] that the $\mathcal{N}=4$, $d=4$, $\mathrm{SU}(N_c)$ super Yang-Mills theory is dual to IIB supergravity with spacetime geometry $\mathrm{AdS}_5 \times \mathbb{S}^5$.

If you know nothing about supersymmetry, it is okay (I myself only have a vague notion of it, as well). The important result we want to quote is

$$\mathrm{CFT}_d \Leftrightarrow \text{(classical) gravity on } \mathrm{AdS}_{d+1} \times \mathcal{Y}$$

with $\mathcal{Y}$ some compact space which is required for a consistent embedding into string theory. It doesn't matter for our purpose today.

In the parenthesis, we mentioned the gravity can be *classical*. This can be achieved for certain CFTs. According to the duality, the parameters on two sides match. On the quantum (CFT) side, we have coupling constant λ and effective number of degrees of freedom c_{eff}. The limit we are interested in is $\lambda, c_{\text{eff}} \to \infty$, which according to AdS/CFT correspondence, is dual to classical gravitational theory. (Effective Planck constant $\hbar \sim 1/c_{\text{eff}}$ and string length $l_s \sim \lambda^{-1/2}$.)

To proceed, we only limit our scope to a CFT_d on background spacetime $\mathcal{B}_d$ which is globally hyperbolic and has parameters $\lambda, c_{\text{eff}} \gg 1$ to have a classical gravity dual. Also, we limit our interest only in the states (said to belong to *code subspace*) on $\mathcal{B}_d$ that can be described geometrically in terms of a *bulk* spacetime $\mathcal{M}_{d+1}$ with $\partial\mathcal{M}_{d+1} = \mathcal{B}_d$. The bulk spacetime has action

$$S_{\text{bulk}} = \frac{1}{16\pi G_N^{(d+1)}} \int \mathrm{d}^{d+1}x \sqrt{-g}\left(R - 2\Lambda + \mathcal{L}_{\text{matter}}\right),$$

leading to Einstein equations

$$G_{ab} + \Lambda g_{ab} = T_{ab}^{\text{matter}}$$

with boundary condition $\partial\mathcal{M}_{d+1} = \mathcal{B}_d$.

Vacuum states in CFT_d are dual to vacuum AdS_{d+1} spaces as the matched symmetries are preserved, for example:

- For Einstein static universe $\mathcal{B}_d = \mathbb{R} \times \mathbb{S}^{d-1}$, the bulk spacetime is the global AdS_{d+1} geometry with metric

$$\mathrm{d}s^2 = -\left(1 + \frac{\rho^2}{l_{\text{AdS}}^2}\right)\mathrm{d}t^2 + \left(1 + \frac{\rho^2}{l_{\text{AdS}}^2}\right)^{-1}\mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\Omega_{d-1}^2\,.$$

- For Minkowski spacetime $\mathcal{B}_d = \mathbb{R}^{d-1,1}$, the bulk spacetime is the Poincaré patch of AdS_{d+1} with metric

$$\mathrm{d}s^2 = \frac{l_{\text{AdS}}^2}{z^2}\left(-\,\mathrm{d}t^2 + \mathrm{d}\mathbf{x}_{d-1}^2 + \mathrm{d}z^2\right).$$

For *excited* states, the story is a bit different. By holographic principle, we have a *dictionary* that matches local operators in CFT_d on $\mathcal{B}_d$ with local matter fields in the bulk $\mathcal{M}_{d+1}$. The matter field back-reacts to the spacetime and changes the geometry. In this case the states are matched to non-trivial asymptotically AdS_{d+1} which can be obtained by solving Einstein equations.

Also, a powerful result to use later is

$$\mathcal{Z}_{\text{CFT}} = \mathcal{Z}_{\text{gravity}}$$

i.e. the partition functions of the two theories agree.

If you are interested in this specific subject, I recommend the Gauge/Gravity Duality lectures in Easter and some good reviews [3, 4, 5, 6].

4 Ryu-Takayanagi Prescription

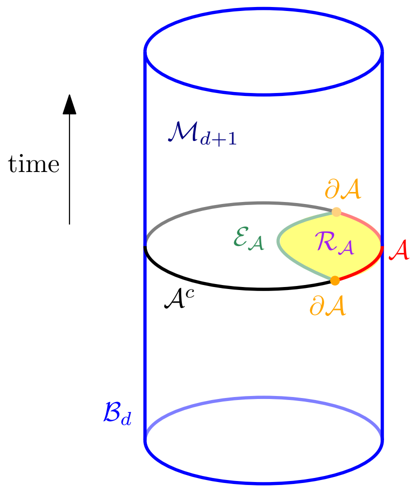
Now we know enough about AdS/CFT correspondence to qualitatively understand how we can calculate entanglement entropy in $\mathcal{B}_d$ via holographic methods. Such question was first addressed by Ryu and Takayanagi (RT) in [7, 8], in which they gave a prescription for static time-independent situations. In this talk we only cover this time-independent case.

Given a holographic CFT_d on background spacetime $\mathcal{B}_d$, we want to compute the entanglement entropy of a given spatial region $\mathcal{A}$ on some Cauchy slice $\Sigma_{d-1} \subset \mathcal{B}_d$. As we mentioned before, we consider $\Sigma_{d-1} = \mathcal{A} \cup \mathcal{A}^c$ and we call $\partial\mathcal{A}$ the *entangling surface*.

The RT prescription for such computation is very easy to state. We need to find a surface $\mathcal{E}_\mathcal{A}$ which

1. is an extremal surface (local extremum of the area functional),
2. is a codimension-2 in the bulk spacetime $\mathcal{M}_{d+1}$,
3. is anchored on $\partial\mathcal{A}$, i.e. $\mathcal{E}_\mathcal{A}|_{\mathcal{B}_d} = \partial\mathcal{A}$,
4. satisfies a *homology constraint*, namely $\mathcal{E}_\mathcal{A}$ is smoothly retractable to the boundary region $\mathcal{A}$, or, in other words, there exists a spacelike, bulk codimension-1, smooth interpolating surface $\mathcal{R}_\mathcal{A} \subset \mathcal{M}_{d+1}$ bounded by $\mathcal{E}_\mathcal{A}$ and $\mathcal{A}$.

This can be demonstrated by the following figure.

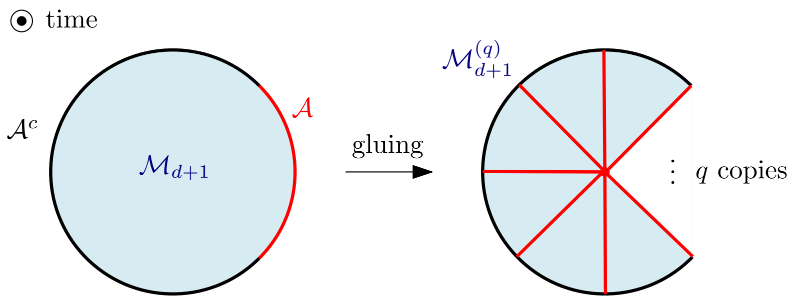


From all available $\mathcal{E}_\mathcal{A}$, we pick the one with the *smallest* area. Then the entanglement entropy can be given in terms of this smallest area. To wit,

$$S_\mathcal{A} = \min_{\mathcal{E}_\mathcal{A}} \frac{\text{Area}(\mathcal{E}_\mathcal{A})}{4G_N^{(d+1)}}, \quad \mathcal{E}_\mathcal{A} : \begin{cases} \partial\mathcal{E}_\mathcal{A} \equiv \mathcal{E}_\mathcal{A}|_{\partial\mathcal{M}_{d+1}} = \partial\mathcal{A}, \\ \exists\mathcal{R}_\mathcal{A} \subset \mathcal{M}_{d+1} : \partial\mathcal{R}_\mathcal{A} = \mathcal{E}_\mathcal{A} \cup \mathcal{A}. \end{cases}$$

The derivation of this prescription is technical and we don't have enough time to show it completely. We only give a very qualitative, sketchy outline.

1. To calculate the Rényi entropy, we used replica trick to 'glue' q copies of $\mathcal{B}_d$ together to get manifold $\mathcal{B}_d^{(q)}$. Now we take the dual bulk $\mathcal{M}_{d+1}^{(q)}$ which also preserve the $\mathbb{Z}_q$ symmetry such that its boundary is $\mathcal{B}_d^{(q)}$.



2. We are in the classical limit in the bulk, so the partition function path integral can be approximated by saddle point, i.e.

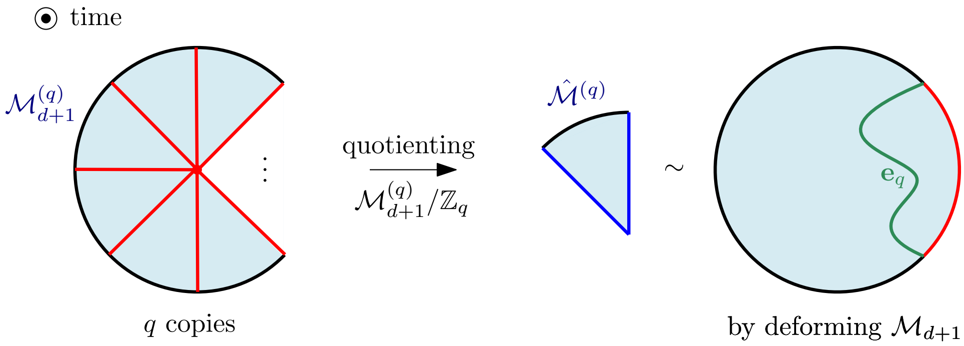
$$\mathcal{Z}_\text{gravity}[\mathcal{M}_{d+1}^{(q)}] \approx e^{-I[\mathcal{M}_{d+1}^{(q)}]}$$

   where $I[\mathcal{M}_{d+1}^{(q)}]$ means *on-shell* action.

3. As the partition function $\mathcal{B}_d$ and $\mathcal{M}_{d+1}$ match, the Rényi entropy is given by the on-shell actions in the classical bulk gravity

$$\begin{aligned} S^{(q)} &= \frac{1}{1-q}\left(\log\mathcal{Z}_\text{gravity}[\mathcal{M}_{d+1}^{(q)}] - q\log\mathcal{Z}_\text{gravity}[\mathcal{M}_{d+1}^{(1)}]\right) \\ &\approx \frac{1}{q-1}\left(I[\mathcal{M}_{d+1}^{(q)}] - qI[\mathcal{M}_{d+1}^{(1)}]\right). \end{aligned}$$

4. We can take the quotient $\hat{\mathcal{M}}^{(q)} = \mathcal{M}_{d+1}^{(q)}/\mathbb{Z}_q$, which is known to be an *orbifold*. It should be topologically equivalent to $\mathcal{M}_{d+1}$ despite some subtleties. Such structure should have a bulk codimension-2 *singular* locus $\mathbf{e}_q$ as we obtained $\mathcal{M}_{d+1}^{(q)}$ by gluing. Such singular locus behave like conical defects and we can see it as some matter field back-reacting on $\mathcal{M}_{d+1}$.



Also, it can be shown that

$$I[\mathcal{M}_{d+1}^{(q)}] = qI[\hat{\mathcal{M}}^{(q)}]$$

so

$$S = \lim_{q\to 1} S^{(q)} = \left(q\partial_q I[\hat{\mathcal{M}}^{(q)}] \right) \Big|_{q=1}.$$

5. It can be shown that

$$\lim_{q\to 1} \mathbf{e}_q \to \mathcal{E}_\mathcal{A}$$

i.e. the singular locus behaves as the desired extremal surface in the limit $q \to 1$. Also, by investigating the geometry near $\mathbf{e}_q$, the calculation gives

$$\partial_q I[\hat{\mathcal{M}}^{(q)}] = \frac{\text{Area}(\mathbf{e}_q)}{4q^2 G_N^{(d+1)}}$$

and we're done.

The generalisations of RT prescription include Hubeny-Rangamani-Takayanagi (HRT) prescription [9] and Aron Wall's reformulation of HRT by a maximin construction [10]. We don't have time to elaborate on these. If interested, please refer to the original papers.

5 Applications in Black Hole Thermodynamics

Finally, we very briefly discuss the possible applications of holographic entanglement entropy in black hole thermodynamics. As we've introduced, CFT with no thermal behaviour is dual to AdS space. A remarkable result is that

CFT with finite temperature $\Leftrightarrow$ black holes.

Also, it is suggested that under such setup, the black hole entropy is equal to the entanglement entropy in the CFT. Recent work [11] has shown remarkable holographic resolution of JKM ambiguity of black hole entropy in higher curvature theories, which is then confirmed by a pure classical method [12].

– THE END –

References

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