UNIVERSITY OF CAMBRIDGE MATHEMATICAL TRIPOS

Part II – **DiffGeo for General Relativity**

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This set of notes is only a chapter (Chapter B) in the Part II General Relativity course taught by Dr. U. Sperhake at University of Cambridge. It basically follows the original notes, yet I might have modified many thing and added some comments. The main purpose is to revise the mathematical framework that underlies General Relativity.

Visit: `http://www.damtp.cam.ac.uk/user/us248/Lectures/lectures.html` for more.

Introduction

GR generalises SR like Riemannian geometry generalises Euclidean geometry.

Conventions. Upstairs (contravariant) index in denominator counts as a downstairs (covariant) component. (e.g. $\partial_i = \frac{\partial}{\partial x^i}$)

1 Manifolds and Tensors

1.1 Manifold

DEFINITION 1.1. An n dimensional manifold $\mathcal{M}$ is a set of point such that there exist a one-to-one map

$$\phi : \mathcal{M} \to U \subset \mathbb{R}^n, \quad p \mapsto x^\alpha, \quad \alpha = 0, 1, \cdots, n-1$$

where $U \subset \mathbb{R}^n$ is open. x^α are the coordinates of the point.

COMMENT BY NOTE TAKER. I think this definition is wrong. The one-to-one map should exist only locally w.r.t. some points rather than globally. (If globally, why bother defining a manifold? Just use $\mathbb{R}^n$!)

COMMENT.

1) It is sufficient if we can chop up $\mathcal{M}$ and map each part to a subset of $\mathbb{R}^n$.

2) Think of coordinates as house numbers.

3) Curves, vectors, etc. live on the $\mathcal{M}$, but ϕ is one-to-one, so this distinction becomes blurred.

1.2 Functions and Curves

DEFINITION 1.2. A function on $\mathcal{M}$ is a map $f : \mathcal{M} \to \mathbb{R}$. f is smooth iff for all coordinate systems x^α, $f(x^\alpha)$ is a smooth function from $\mathbb{R}^n$ to $\mathbb{R}$. If f is invariant under coordinate transformation, it is called a scalar.

DEFINITION 1.3. A curve on $\mathcal{M}$ is a map $\lambda : I \subset \mathbb{R} \to \mathcal{M}$, I is open. λ is smooth iff for all coordinate system x^α, the map $x^\alpha \circ \lambda : I \to \mathbb{R}$ is smooth.

DEFINITION 1.4. $\mathcal{C}^\infty$ is the space of all smooth functions $f : \mathcal{M} \to \mathbb{R}$.

1.3 Vectors

DEFINITION 1.5. Let λ be a smooth curve $\lambda : I \subset \mathbb{R} \to \mathcal{M}$ with $p = \lambda(0)$. The tangent vector to the curve λ at $p \in \mathcal{M}$ is the map

$$V : \mathcal{C}^\infty \to \mathbb{R}, \quad f \mapsto V(f) = \frac{\mathrm{d}}{\mathrm{d}t} f(\lambda(t))\Big|_{t=0}.$$

$T_p(\mathcal{M})$ is the space of all vectors at p, i.e. the tangent space of manifold $\mathcal{M}$ at point p.

That is to say a vector is a derivative operator.

PROPOSITION 1.1.

(i) *Linearity: for* $\alpha, \beta \in \mathbb{R}$, $f, g, \in \mathcal{C}^\infty$, *we have* $V(\alpha f + \beta g) = \alpha V(f) + \beta V(g)$

(ii) *Leibnitz: for* $f, g \in \mathcal{C}^\infty$, *we have* $V(fg) = V(f)g(p) + f(p)V(g)$

In coordinates x^α, we can write

$$V(f) = \frac{\mathrm{d}}{\mathrm{d}t} f(x^\mu(\lambda(t))) = \underbrace{\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\Big|_\lambda}_{\text{components}} \underbrace{\frac{\partial}{\partial x^\mu}}_{\text{basis}} f(x^\alpha)$$

One can show $T_p(\mathcal{M})$ has dimension n and

$$e_\mu := \partial_\mu = \frac{\partial}{\partial x^\mu}$$

form a basis. The components are

$$V^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}t}.$$

We can write

$$\boxed{V = V^\mu e_\mu = V^\mu \partial_\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\partial}{\partial x^\mu} = \frac{\mathrm{d}}{\mathrm{d}t}}$$

As for coordinate change $x^\mu \to \tilde{x}^\alpha$,

$$\boxed{\tilde{e}_\alpha = \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} e_\mu}$$

using chain rule.

Similarly, we can find the components under coordinate transformation is

$$\boxed{\tilde{V}^\alpha = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} V^\mu}$$

Overall,

$$V = V^\mu e_\mu = \tilde{V}^\alpha \tilde{e}_\alpha = \tilde{V}$$

is invariant.

∂_μ is called a “coordinate basis”. There also exists non-coordinate basis.

1.4 Covectors/One-Forms

DEFINITION 1.6. A covector or one-form is a linear map

$$\eta : T_p(\mathcal{M}) \to \mathbb{R}, \quad V \mapsto \eta(V).$$

Write $T_p^*(\mathcal{M})$ as the space of all covectors at $p \in \mathcal{M}$, called the cotangent space. $T_p^*(\mathcal{M})$ is an n-dimensional vector space. Let e_μ be a basis of $T_p(\mathcal{M})$. The components of η are $\eta_\mu := \eta(e_\mu)$.

PROPOSITION 1.2.

(i) Linearity: for $\alpha, \beta \in \mathbb{R},\ V, W \in T_p(\mathcal{M})$, *we have* $\eta(\alpha V + \beta W) = \alpha\eta(V) + \beta\eta(W)$

(ii) Components: $\eta(V) = \eta(V^\mu e_\mu) = V^\mu \eta_\mu$

(iii) Transformation: We require $\eta(V)$ *to be a scalar, this means*

$$\eta(V) = \eta_\mu V^\mu = \tilde{\eta}_\alpha \tilde{V}^\alpha = \tilde{\eta}_\alpha \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} V^\mu$$

i.e.

$$\boxed{\tilde{\eta}_\beta = \frac{\partial x^\mu}{\partial \tilde{x}^\beta} \eta_\mu}$$

DEFINITION 1.7. The gradient $\mathrm{d}f$ of a smooth function f is a map

$$\mathrm{d}f : T_p(\mathcal{M}) \to \mathbb{R}, \quad V = \frac{\mathrm{d}}{\mathrm{d}t} \mapsto \frac{\mathrm{d}f}{\mathrm{d}t} = V(f).$$

It can be used as a basis. Let $f = x^\alpha$ with α fixed. Then

$$\mathrm{d}x^\alpha(e_\beta) = \mathrm{d}x^\alpha \left(\frac{\partial}{\partial x^\beta} \right) = \frac{\partial x^\alpha}{\partial x^\beta} = \delta^\alpha{}_\beta.$$

This means

$$\eta_\alpha \mathrm{d}x^\alpha(V) = \eta_\alpha \mathrm{d}x^\alpha(V^\beta \partial_\beta) = \eta_\alpha V^\beta \delta^\alpha{}_\beta = \eta_\alpha V^\alpha$$

i.e.

$$\boxed{\eta = \eta_\alpha \mathrm{d}x^\alpha}$$

That is to say $\mathrm{d}x^\alpha$ is the coordinate basis of $T_p^*(\mathcal{M})$.

1.5 Tensors

DEFINITION 1.8. A tensor T at $p \in \mathcal{M}$ of rank $\binom{r}{s}$, $r, s \in \mathbb{N}_0$, is a multilinear map

$$T : \underbrace{T_p^*(\mathcal{M}) \times \cdots \times T_p^*(\mathcal{M})}_{r \text{ copies}} \times \underbrace{T_p(\mathcal{M}) \times \cdots \times T_p(\mathcal{M})}_{s \text{ copies}} \to \mathbb{R}$$

to which we plug in r one-forms and s vectors to get a real number.

EXAMPLE.

- Any covector η is a tensor of rank $\binom{0}{1}$.
- Any vector V is a tensor of rank $\binom{1}{0}$.
- For a tensor $\delta : T_p^*(\mathcal{M}) \times T_p(\mathcal{M}) \to \mathbb{R}$, $(\eta, V) \mapsto \eta(V)$, $\forall \eta \in T_p^*(\mathcal{M})$, $V \in T_p(\mathcal{M})$ is a tensor of rank $\binom{1}{1}$ with components $\delta^\alpha{}_\beta = \delta(\mathrm{d}x^\alpha, \partial_\beta) = \partial x^\alpha / \partial x^\beta =$ Kronecker Delta.

COMMENT BY NOTE TAKER. Consider this:

$$\eta(V) = \eta_\alpha \mathrm{d}x^\alpha(V) = \eta_\alpha V^\alpha \Rightarrow V^\alpha = \mathrm{d}x^\alpha(V) \overset{!}{=} V(\mathrm{d}x^\alpha).$$

Actually we want to show that there is an isomorphism between $\mathcal{C}^\infty$ and $T_p^*(\mathcal{M})$ so that we can write vectors as maps $T_p^*(\mathcal{M}) \to \mathbb{R}$. This may be shown by studying the map $\mathrm{d} : \mathcal{C}^\infty \to T_p^*(\mathcal{M})$, $f \mapsto \mathrm{d}f$. However, I haven't formulated the proof myself yet.

As for transformation of $\binom{r}{s}$ tensor, one can show

$$\boxed{\tilde{T}^{\alpha_1 \cdots \alpha_r}{}_{\beta_1 \cdots \beta_s} = \frac{\partial \tilde{x}^{\alpha_1}}{\partial x^{\mu_1}} \cdots \frac{\partial \tilde{x}^{\alpha_r}}{\partial x^{\mu_r}} \frac{\partial x^{\nu_1}}{\partial \tilde{x}^{\beta_1}} \cdots \frac{\partial x^{\nu_s}}{\partial \tilde{x}^{\beta_s}} T^{\mu_1 \cdots \mu_r}{}_{\nu_1 \cdots \nu_s}}$$

1.6 Tensor Operations

(1) Addition and scalar multiplication: Let $c_1, c_2 \in \mathbb{R}$, S, T are $\binom{1}{1}$ tensors, we have

$$c_1 S + c_2 T : T_p^*(\mathcal{M}) \times T_p(\mathcal{M}) \to \mathbb{R}, \quad (\eta, V) \mapsto c_1 S(\eta, V) + c_2 T(\eta, V).$$

(2) (Anti-)Symmetrization: E.g. for $\binom{0}{2}$ tensor T,

  (a) Symmetric part $S_{\alpha\beta} := \frac{1}{2}(T_{\alpha\beta} + T_{\beta\alpha}) =: T_{(\alpha\beta)}$

  (b) Anti-symmetric part $A_{\alpha\beta} := \frac{1}{2}(T_{\alpha\beta} - T_{\beta\alpha}) =: T_{[\alpha\beta]}$

  (c) Index subset: e.g. $T^{(\alpha\beta)\gamma}{}_\delta := \frac{1}{2}(T^{\alpha\beta\gamma}{}_\delta + T^{\beta\alpha\gamma}{}_\delta)$

  (d) Non-adjacent indices: e.g. $T_{(\alpha|\beta\gamma|\delta)} := \frac{1}{2}(T_{\alpha\beta\gamma\delta} + T_{\delta\beta\gamma\alpha})$

  (e) Over $n > 2$ indices, sum over all permutations. Sign of permutation should be applied for anti-symmetrization. A factor of $n!$ should be divided by. E.g.

$$T^\alpha{}_{[\beta\gamma\delta]} = \frac{1}{3!}(T^\alpha{}_{\beta\gamma\delta} + T^\alpha{}_{\gamma\delta\beta} + T^\alpha{}_{\delta\beta\gamma} - T^\alpha{}_{\gamma\beta\delta} - T^\alpha{}_{\delta\gamma\beta} - T^\alpha{}_{\beta\delta\gamma})$$

(3) Contraction of $\binom{r}{s}$ tensor: Summation over one upstairs and one downstairs index.

EXAMPLE. Let T be a $\binom{3}{2}$ tensor. It can give a $\binom{2}{1}$ tensor S by defining

$$S(\omega, \eta, V) := T(\mathrm{d}x^\mu, \omega, \eta, \partial_\mu, V) \quad \text{(sum over } \mu)$$

This is basis independent since $\mathrm{d}x^\alpha(\partial_\beta) = \delta^\alpha{}_\beta$, $\mathrm{d}\tilde{x}^\mu(\tilde{\partial}_\nu) = \delta^\mu{}_\nu$ and $\mathrm{d}\tilde{x}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\alpha}\mathrm{d}x^\alpha$, we have

$$T(\mathrm{d}\tilde{x}^\mu, \omega, \eta, \tilde{\partial}_\mu, V) = \underbrace{\frac{\partial \tilde{x}^\mu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial \tilde{x}^\mu}}_{\delta^\beta{}_\alpha} T(\mathrm{d}x^\alpha, \omega, \eta, \partial_\beta, V)$$

is a tensor. The components are

$$S^{\mu\nu}{}_\rho = T^{\alpha\mu\nu}{}_{\alpha\rho}.$$

(4) Outer product: Let S be a $\binom{p}{q}$ tensor and T a $\binom{r}{s}$ tensor. The outer product $S \otimes T$ is a $\binom{p+r}{q+s}$ tensor.

$$(S \otimes T)(\omega_1, \cdots, \omega_p, \eta_1, \cdots, \eta_r, X_1, \cdots, X_q, Y_1, \cdots, Y_s) := S(\omega_1, \cdots, \omega_p, X_1, \cdots, X_q) T(\eta_1, \cdots, \eta_r, Y_1, \cdots, Y_s)$$

One shows

(i) $(S \otimes T)^{\alpha_1 \cdots \alpha_p \beta_1 \cdots \beta_r}{}_{\mu_1 \cdots \mu_q \nu_1 \cdots \nu_s} = S^{\alpha_1 \cdots \alpha_p}{}_{\mu_1 \cdots \mu_q} T^{\beta_1 \cdots \beta_r}{}_{\nu_1 \cdots \nu_s}$

(ii) E.g. in a coordinate basis, a $\binom{2}{1}$ tensor T can be written $T = T^{\mu\nu}{}_\rho\,(e_\mu \otimes e_\nu \otimes \mathrm{d}x^\rho)$ where $e_\mu = \partial_\mu$. Likewise for $\binom{r}{s}$ tensor.

1.7 Tensor Field

DEFINITION 1.9. A tensor field of rank $\binom{r}{s}$ is a collection of $\binom{r}{s}$ tensors at each point $p \in \mathcal{M}$. Like a map $p \mapsto T_p$ of rank $\binom{r}{s}$. The tensor field is smooth iff its components in any coordinate basis are smooth functions.

Some times we denote X_p as a vector but X as a field.

EXAMPLE. Vector field $X : \mathcal{M} \to T_p(\mathcal{M})$, $p \mapsto X_p$.

For a function f, $X(f) : \mathcal{M} \to \mathbb{R}$, $p \mapsto X_p(f)$ is a function.

1.8 Integral Curves

DEFINITION 1.10. Integral curve λ of a vector field V through a point $p \in \mathcal{M}$ is the curve through p whose tangent vector at every point q along the curve is V_q.

We know that $\frac{\mathrm{d}}{\mathrm{d}t}\big|_\lambda = V$, in coordinate basis, the equation

$$\frac{\mathrm{d}x^\mu(\lambda(t_0))}{\mathrm{d}t} = V^\mu(x^\nu)$$

with $x^\mu(\lambda(t_0)) = x^\mu(p)$ has a unique solution by ODE theory.

2 The Metric Tensor

2.1 Metric Tensor

DEFINITION 2.1. A metric at $p \in \mathcal{M}$ is a $\binom{0}{2}$ tensor g that is

(i) Symmetric: $g(V,W) = g(W,V),\ \forall\ V,W \in T_p(\mathcal{M})$, i.e. $g_{\alpha\beta} = g_{\beta\alpha}$;

(ii) Non-degenerate $g(V,W) = 0,\ \forall\ W \in T_p(\mathcal{M}) \Leftrightarrow V = 0$.

The components are $g = g_{\alpha\beta}\mathrm{d}x^\alpha \otimes \mathrm{d}x^\beta$, or $g_{\mu\nu} = g(\partial_\mu, \partial_\nu)$.

COMMENT BY NOTE TAKER. The metric is used to describe the distance between two infinitely close points on the manifold, which is an intrinsic property of the manifold. Denote

$$\mathrm{d}s^2 = g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta.$$

Note that $\mathrm{d}x^\mu$ here is *not* a one-form. It is simply the differential of coordinate x^μ.

A metric maps vectors to one-forms by $V \mapsto g(V,\cdot) =: \underline{V}$, i.e.

$$\underline{V} : T_p(\mathcal{M}) \to \mathbb{R}, \quad W \mapsto \underline{V}(W) := g(V,W) = \underline{V}_\nu W^\nu = g_{\mu\nu}V^\mu W^\nu.$$

We denote components of $\underline{V}$

$$V_\mu := \underline{V}_\mu = g_{\mu\nu}V^\nu.$$

As g is non-degenerate, it is invertible.

DEFINITION 2.2. g^{-1} is the inverse metric. It is a symmetric $\binom{2}{0}$ tensor. The components are $g^{\alpha\beta}$ with $g^{\alpha\beta}g_{\beta\gamma} = \delta^a{}_\gamma$.

EXAMPLE. Line element on the unit sphere $x^2+y^2+z^2 = 1$ in $\mathbb{R}^3$ is $\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$, then the metric and inverse metric are

$$[g_{\alpha\beta}] = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \qquad [g^{\alpha\beta}] = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2\theta} \end{pmatrix}.$$

g^{-1} maps one-forms to vectors by $\eta \mapsto g^{-1}(\eta,\cdot) =: \overline{\eta}$, such that $\overline{\eta}(\omega) = g^{-1}(\eta,\omega)$. The components are

$$\eta^\alpha := \overline{\eta}^\alpha = g^{\alpha\beta}\eta_\beta.$$

The metric mappings between vectors and one-forms are inverse of each other, i.e.

$$g^{-1}(g(V,\cdot),\cdot) = V; \qquad g(g^{-1}(\eta,\cdot),\cdot) = \eta.$$

This establishes a natural isomorphism between vectors and covectors/one-forms.

2.2 Signature

Since g is symmetric, the components $g_{\alpha\beta}$ at $p \in \mathcal{M}$ are a symmetric matrix. Therefore, there exists a basis where $g_{\mu\nu}$ is diagonal. By the fact that g is non-degenerate, all diagonal values of the matrix are non-zero. We can rescale such that all diagonal elements are $+1$ and -1.

Law 2.1 (Sylvester's Law). *The number of* $+1, -1$ *is independent of how we choose orthonormal basis.*

Definition 2.3. The signature is the sum of $+1, -1$ over all diagonal elements.

For Riemannian metric: the signature is $+\cdots+ = +n =$ number of dimensions.

For Lorentzian metric: the signature is $-+\cdots+ = n-2$. (The other convention is $+-\cdots- = 2-n$.)

Note. Equivalence principle suggests that in a local inertial frame, the laws of SR hold. That is to say there exist coordinates such that $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$. This is Lorentz invariant, only possible locally. At some other points, $g_{\mu\nu} \neq \eta_{\mu\nu}$ in general.

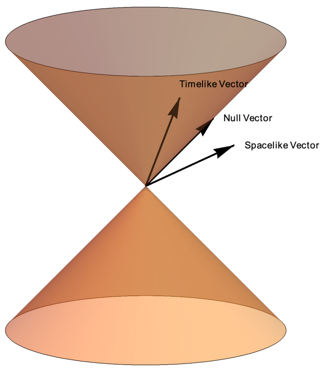
Definition 2.4. A Riemannian (Lorentzian) manifold is $(\mathcal{M}, g)$ where $\mathcal{M}$ is a differential manifold and g is a Riemannian (Lorentzian) metric.

Example. Minkowski metric η in $\mathbb{R}^4$ with Cartesian coordinates x^0, x^1, x^2, x^3:

$$\eta = -(\mathrm{d}x^0 \otimes \mathrm{d}x^0) + (\mathrm{d}x^1 \otimes \mathrm{d}x^1) + (\mathrm{d}x^2 \otimes \mathrm{d}x^2) + (\mathrm{d}x^3 \otimes \mathrm{d}x^3).$$

Definition 2.5. Let $(\mathcal{M}, g)$ be a Lorentzian manifold, for $V \in T_p(\mathcal{M}), V \neq 0$, say V is timelike iff $g(V,V) < 0$; V is null iff $g(V,V) = 0$; V is spacelike iff $g(V,V) > 0$.

In local inertial frame, $g_{\mu\nu} = \eta_{\mu\nu}$, locally we have the light cone structure of SR.



Definition 2.6. Spacetime is a four dimensional Lorentzian manifold.

Definition 2.7. The norm of spacelike vector V is

$$|V| := \sqrt{g(V,V)}.$$

The angle between spacelike vectors V, W is

$$\cos\theta := \frac{g(V,W)}{|V||W|}.$$

3 Geodesics

3.1 Curves

Definition 3.1. A curve is timelike (null, spacelike) at a point $p \in \mathcal{M}$ iff its tangent vector at p is timelike (null, spacelike).

Note. This can change along a curve.

Definition 3.2. Length along a spacelike curve is

$$s := \int_{t_1}^{t_2} \sqrt{g(V,V)\Big|_{\lambda(t)}}\,\mathrm{d}t = \int_{t_1}^{t_2} \sqrt{g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\mathrm{d}x^\beta}{\mathrm{d}t}}\,\mathrm{d}t\,, \quad V = \frac{\mathrm{d}}{\mathrm{d}t}.$$

Proper time along a timelike curve $\lambda(t)$ is

$$\tau := \int_{t_1}^{t_2} \sqrt{-\,g(V,V)\Big|_{\lambda(t)}}\,\mathrm{d}t = \int_{t_1}^{t_2} \sqrt{-g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\mathrm{d}x^\beta}{\mathrm{d}t}}\,\mathrm{d}t\,.$$

Definition 3.3. Four velocity along a timelike curve is

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}t}\Big|_{\lambda(t)}.$$

By this definition, we have

$$g_{\mu\nu}u^\mu u^\nu = -1.$$

3.2 Noether's Theorem

Action

$$\mathcal{S} = \int \mathcal{L}(q_k, \dot{q}_k, \lambda)\,\mathrm{d}\lambda$$

is extremised by the curve satisfying the Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_k}\right) = \frac{\partial\mathcal{L}}{\partial q_k}$$

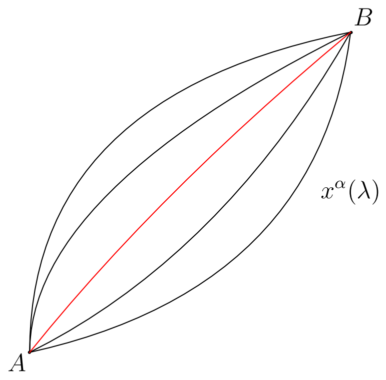
Theorem 3.1 (Noether's Theorem).

(i) If $\mathcal{L}$ is not explicitly dependent on q_k, the quantity $p_k := \frac{\partial\mathcal{L}}{\partial\dot{q}_k}$ is conserved along curve that extremises $\mathcal{S}$.

(ii) If $\mathcal{L}$ is not explicitly dependent on λ, the quantity $I := \dot{q}_k\frac{\partial\mathcal{L}}{\partial\dot{q}_k} - \mathcal{L}$ is conserved along the curve that extremises $\mathcal{S}$.

3.3 Geodesics, Variation 1

For a curve from A to B, wlog, we choose $\lambda = 0$ at A and $\lambda = 1$ at B.



The action and Lagrangian are

$$\mathcal{S} = \int_0^1 \mathcal{L}\,\mathrm{d}\lambda\,, \quad \mathcal{L} = \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$$

where $\dot{x}^\mu = \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}$.

We discuss timelike curves now.

NOTE. $\mathcal{S}$ is invariant under change of parameter,

$$\lambda \to \kappa(\lambda),\ \frac{\mathrm{d}\kappa}{\mathrm{d}\lambda} > 0 : \int_0^1 \sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\,\mathrm{d}\lambda = \int_{\kappa(0)}^{\kappa(1)} \sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\kappa}\frac{\mathrm{d}x^\nu}{\mathrm{d}\kappa}}\,\mathrm{d}\kappa\,.$$

To calculate EL equations:

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} = \frac{1}{2\mathcal{L}}(-g_{\mu\nu}\delta^\mu{}_\alpha \dot{x}^\nu - g_{\mu\nu}\dot{x}^\mu \delta^\nu{}_\alpha) = -g_{\mu\alpha}\frac{\dot{x}^\mu}{\mathcal{L}}$$

$$\frac{\partial \mathcal{L}}{\partial x^\alpha} = \frac{1}{2\mathcal{L}}(-\dot{x}^\mu \dot{x}^\nu \partial_\alpha g_{\mu\nu})$$

Thus the EL equations are

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{-g_{\mu\alpha}\dot{x}^\mu}{\mathcal{L}}\right) + \frac{\dot{x}^\mu \dot{x}^\nu}{2\mathcal{L}}\partial_\alpha g_{\mu\nu} = 0$$

Change parameter

$$\tau(\lambda) = \int_0^\lambda \sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tilde{\lambda}}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tilde{\lambda}}}\,\mathrm{d}\tilde{\lambda}$$

we have

$$\frac{\mathrm{d}\tau}{\mathrm{d}\lambda} = \mathcal{L} \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\lambda} = \frac{\mathrm{d}\tau}{\mathrm{d}\lambda}\frac{\mathrm{d}}{\mathrm{d}\tau} = \mathcal{L}\frac{\mathrm{d}}{\mathrm{d}\tau}.$$

Then

$$-\mathcal{L}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{\mu\alpha}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\right) + \frac{\mathcal{L}}{2}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\partial_\alpha g_{\mu\nu} = 0$$

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2}g_{\mu\alpha} + \partial_\nu g_{\mu\alpha}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} - \frac{1}{2}\partial_\alpha g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 0$$

Multiply both sides with $g^{\beta\alpha}$, we have

$$\boxed{\frac{\mathrm{d}^2 x^\beta}{\mathrm{d}\tau^2} + \left\{ {}^{\ \beta}_{\mu\ \nu} \right\}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 0} \tag{1}$$

DEFINITION 3.4. The Christoffel symbols are

$$\boxed{\left\{ {}_{\mu}{}^{\beta}{}_{\nu} \right\} := \frac{1}{2} g^{\rho\beta} (\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})}$$

For spacelike geodesics:

$$\tilde{\mathcal{L}} = \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}, \quad \frac{\mathrm{d}s}{\mathrm{d}\lambda} = \tilde{\mathcal{L}}.$$

This just gives the same equation with $\tau \to s$.

3.4 Geodesics, Variation 2

Alternatively, we can have the action and the Lagrangian as

$$\hat{\mathcal{S}} = \int_A^B \hat{\mathcal{L}} \,\mathrm{d}\lambda\,, \quad \hat{\mathcal{L}} = g_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}$$

This has no restriction to timelike geodesics, but $\hat{\mathcal{S}}$ is not invariant under reparametrisation.

The EL equations are

$$\ddot{x}^\alpha - \left\{ {}_{\nu}{}^{\alpha}{}_{\beta} \right\} \dot{x}^\nu \dot{x}^\beta = 0 \tag{2}$$

where $\dot{} \equiv \frac{\mathrm{d}}{\mathrm{d}\lambda}$.

But now take equation 1 and let $\tau = \tau(\lambda)$, $\frac{\mathrm{d}\tau}{\mathrm{d}\lambda} > 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}}{\mathrm{d}\lambda}, \quad \frac{\mathrm{d}^2}{\mathrm{d}\tau^2} = \frac{\mathrm{d}^2\lambda}{\mathrm{d}\tau^2}\frac{\mathrm{d}}{\mathrm{d}\lambda} + \left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^2 \frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}$$

This gives

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\lambda^2} + \left\{ {}_{\nu}{}^{\alpha}{}_{\beta} \right\} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} = -\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-2} \frac{\mathrm{d}^2\lambda}{\mathrm{d}\tau^2}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} \propto \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} \tag{3}$$

This is not equation 2. The answer to this is that $\hat{\mathcal{S}}$ is not invariant under reparametrisation. Variation of $\hat{\mathcal{S}}$ with different parameter gives a different curve.

Equation 2 and 3 agree if

$$\frac{\mathrm{d}^2\lambda}{\mathrm{d}\tau^2} = 0$$

i.e.

$$\lambda = c_1\tau + c_2, \quad c_1, c_2 \in \mathbb{R}$$

DEFINITION 3.5. The parameter λ along a timelike (spacelike) curve is affine iff it is related linearly to proper time (distance). For a non-affine parameter, the geodesic is equation 3.

For a summary, we have the following definition.

DEFINITION 3.6. If a curve $C : I \subset \mathbb{R} \to \mathcal{M},\ \lambda \mapsto x^\alpha(\lambda)$

(i) satisfies equation 2, it is a geodesic with λ affine;

(ii) satisfies equation 3 with non-zero R.H.S., it is a geodesic with λ not affine;

(iii) satisfies neither equation 2 or 3, it is not a geodesic.

The ODE theory suggests the solutions of equation 2 and 3 are unique if $x^\alpha, \dot{x}^\alpha$ are fixed at $\lambda = \lambda_0$.

3.5 Geodesic Postulate

POSTULATE 1 (Geodesic Postulate). *Test particles with positive (zero) rest mass move on timelike (null) geodesics.*

3.6 Methods of Calculating the Values of Christoffel Symbols

$\hat{\mathcal{L}}$ gives an easy way to calculate $\{ {}_{\nu}{}^{\alpha}{}_{\beta} \}$.

EXAMPLE (Schwarzschild metric). For Schwarzschild metric:

$$\mathrm{d}s^2 = -f\,\mathrm{d}t^2 + f^{-1}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2$$

where

$$f(r) = 1 - \frac{2M}{r}, \quad M = \text{const.}$$

we have the Lagrangian

$$-\hat{\mathcal{L}} = f\dot{t}^2 - f^{-1}\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2, \quad \dot{} \equiv \frac{\mathrm{d}}{\mathrm{d}\tau}$$

The EL equation for $t(\tau)$ is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\big(2f\dot{t}\big) = 0 \quad \Rightarrow \quad \frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} + f^{-1}\frac{\mathrm{d}f}{\mathrm{d}r}\dot{t}\dot{r} = 0$$

this gives

$$\{ {}_{t}{}^{t}{}_{r} \} = \{ {}_{r}{}^{t}{}_{t} \} = \frac{1}{2f}\frac{\mathrm{d}f}{\mathrm{d}r}, \quad \{ {}_{\mu}{}^{t}{}_{\nu} \} = 0 \text{ otherwise.}$$

4 Covariant Derivative

4.1 Covariant Derivative of Scalars and Vectors

DEFINITION 4.1. For some function f, the covariant derivative is

$$\nabla f : T_p(\mathcal{M}) \to \mathbb{R}, \quad V \mapsto \nabla_V f = V(f) = V^\alpha \nabla_\alpha f.$$

∇f is a tensor of rank $\binom{0}{1}$. The components are

$$\nabla_\alpha f = (\nabla f)_\alpha = f_{;\alpha} := \partial_\alpha f.$$

DEFINITION 4.2. For some vector V, the covariant derivative is

$$\nabla V : T_p(\mathcal{M}) \to T_p(\mathcal{M}), \quad X \mapsto \nabla_X V$$

with

(1) $\nabla_{fX+gY} V = f\nabla_X V + g\nabla_Y V$

(2) $\nabla_X(V + W) = \nabla_X V + \nabla_X W$

(3) $\nabla_X(fV) = f\nabla_X V + (\nabla_X f)V$

Equivalently,

$$\nabla V : T_p^*(\mathcal{M}) \times T_p(\mathcal{M}) \to \mathbb{R}, \quad (\eta, X) \mapsto \eta(\nabla_X V)$$

∇V is a $\binom{1}{1}$ tensor with components

$$V^\alpha{}_{;\beta} = \nabla_\beta V^\alpha := (\nabla V)^\alpha{}_\beta$$

DEFINITION 4.3. Let $\{e_\mu\}$ be a basis of $T_p(\mathcal{M})$. The connection coefficients $\Gamma^\rho_{\mu\nu}$ are defined by

$$\nabla_\nu e_\mu = \Gamma^\rho_{\mu\nu} e_\rho.$$

For $V = V^\mu e_\mu, \ W = W^\mu e_\mu$

$$\begin{aligned}
\nabla_V W &= \nabla_V (W^\mu e_\mu) \\
&= V(W^\mu) e_\mu + W^\mu \nabla_v (e_\mu) \\
&= V^\nu e_\nu (W^\mu) e_\mu + W^\mu \nabla_{V^\nu e_\nu} (e_\mu) \\
&= V^\nu e_\nu (W^\mu) e_\mu + W^\mu V^\nu \nabla_{e_\nu} (e_\mu) \\
&= V^\nu (\partial_\nu W^\rho + W^\mu \Gamma^\rho_{\mu\nu}) e_\rho
\end{aligned}$$

Thus

$$(\nabla_V W)^\rho = V^\nu \partial_\nu W^\rho + V^\nu W^\mu \Gamma^\rho_{\mu\nu}$$

Since V is arbitrary,

$$\boxed{W^\rho{}_{;\nu} = \nabla_\nu W^\rho := (\nabla W)^\rho{}_\nu = \partial_\nu W^\rho + \Gamma^\rho_{\mu\nu} W^\mu}$$

Under coordinate transformation $x^\mu \to \tilde{x}^\alpha$, we can show

$$\Gamma'^\sigma_{\mu\nu} = \frac{\partial \tilde{x}^\sigma}{\partial x^\rho} \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\beta}{\partial \tilde{x}^\nu} \Gamma^\rho_{\alpha\beta} + \frac{\partial \tilde{x}^\sigma}{\partial x^\rho} \frac{\partial^2 x^\rho}{\partial \tilde{x}^\mu \partial \tilde{x}^\nu} \tag{4}$$

This means that $\Gamma^\sigma_{\mu\nu}$ is not a tensor, but the difference of two connections is.

DEFINITION 4.4. The torsion tensor is defined as

$$T_{\mu\nu}{}^{\lambda} := \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}.$$

We say Γ is torsion free iff $T_{\mu\nu}{}^{\lambda} = 0$.

COMMENT.

- $\partial_\nu W^\mu = \frac{\partial W^\mu}{\partial x^\nu}$ is not a tensor either;
- The second term of the R.H.S. of equation 4 just cancels this to make $\nabla_\nu W^\mu$ a tensor.

EXAMPLE. $\{ {}^{\ \alpha}_{\beta\ \gamma} \}$ are a connection.

Convention. We use the convention that the derivative index is the 2nd downstairs index.

4.2 Covariant Derivative of Tensors

The covariant derivative of tensors is obtained by requiring Leibnitz rules.

For a rank $\binom{r}{s}$ tensor T, ∇T is a $\binom{r}{s+1}$ tensor.

EXAMPLE. For one-forms,

$$\nabla_V(\eta(W)) := (\nabla_V \eta)(W) + \eta(\nabla_V W)$$

We have

$$\begin{aligned}
(\nabla_V \eta)(W) &= \nabla_V(\eta(W)) - \eta(\nabla_V W) \\
&= V^\rho \partial_\rho(\eta_\mu W^\mu) - \eta_\mu(V^\rho \partial_\rho W^\mu + V^\rho \Gamma^\mu_{\nu\rho} W^\nu) \\
&= V^\rho W^\mu \partial_\rho \eta_\mu - \Gamma^\mu_{\nu\rho} \eta_\mu V^\rho W^\nu \\
&= (\partial_\rho \eta_\mu - \Gamma^\nu_{\mu\rho} \eta_\nu) V^\rho W^\mu
\end{aligned}$$

Finally

$$\boxed{\eta_{\mu;\rho} = \nabla_\rho \eta_\mu = (\nabla \eta)_{\mu\rho} = \partial_\rho \eta_\mu - \Gamma^\nu_{\mu\rho} \eta_\nu}$$

DEFINITION 4.5. For a $\binom{r}{s}$ tensor, the covariant derivative is

$$\begin{aligned}
\nabla_\rho T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s} &= \nabla_\rho T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s} \\
&\underbrace{+\Gamma^{\mu_1}_{\sigma\rho} T^{\sigma\mu_2\cdots\mu_r}{}_{\nu_1\cdots\nu_s} + \cdots + \Gamma^{\mu_r}_{\sigma\rho} T^{\mu_1\cdots\mu_{r-1}\sigma}{}_{\nu_1\cdots\nu_s}}_{\text{for each upstair index}} \\
&\underbrace{-\Gamma^{\sigma}_{\nu_1\rho} T^{\mu_1\cdots\mu_r}{}_{\sigma\nu_2\cdots\nu_s} - \cdots - \Gamma^{\sigma}_{\nu_s\rho} T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_{s-1}\sigma}}_{\text{for each downstair index}}.
\end{aligned}$$

5 The Levi-Civita Connection

NOTE. We need no metric for a connection. However, a metric singles out a special connection.

THEOREM 5.1. *On a manifold $\mathcal{M}$ with a metric g, there exists unique connection with*

(1) $\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu} = \{ {}_\mu{}^\alpha{}_\nu \}$ *(Christoffel symbols, torsion free);*

(2) $\nabla g = 0$ *(metric compatible).*

This is the Levi-Civita connection.

Proof. "$\Rightarrow$": Let $\Gamma^\alpha_{\beta\gamma}$ be metric compatible and symmetric. Then

$$\nabla_\alpha g_{\beta\gamma} = 0 \quad \Rightarrow \quad \partial_\alpha g_{\beta\gamma} = \Gamma^\rho_{\beta\alpha} g_{\rho\gamma} + \Gamma^\rho_{\gamma\alpha} g_{\beta\rho}.$$

By the definition of Christoffel symbols

$$\begin{aligned}
\{ {}_\beta{}^\mu{}_\gamma \} &= \frac{1}{2} g^{\mu\nu} (\partial_\beta g_{\gamma\nu} + \partial_\gamma g_{\nu\beta} - \partial_\nu g_{\beta\gamma}) \\
&= \frac{1}{2} g^{\mu\nu} (\Gamma^\rho_{\gamma\beta} g_{\rho\nu} + \Gamma^\rho_{\nu\beta} g_{\gamma\rho} + \Gamma^\rho_{\nu\gamma} g_{\rho\beta} + \Gamma^\rho_{\beta\gamma} g_{\nu\rho} - \Gamma^\rho_{\beta\nu} g_{\rho\gamma} - \Gamma^\rho_{\gamma\nu} g_{\beta\rho}) \\
&= \Gamma^\mu_{\beta\gamma}
\end{aligned}$$

"$\Leftarrow$": Likewise, set $\Gamma^\alpha_{\beta\gamma} = \{ {}_\beta{}^\alpha{}_\gamma \}$. This suggest the connection is symmetric. $\nabla g = 0$ can be shown. □

In General Relativity, we use Levi-Civita connection.

6 Parallel Transport

DEFINITION 6.1. Let V be a vector field, $\mathcal{C}$ an integral curve of V. A tensor T is parallel transported along $\mathcal{C}$ iff $\nabla_V T = 0$ along the curve.

COMMENT.

- The tangent vector of an affinely parametrised geodesic is parallel transported along itself. Set $U^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}$,

$$\begin{aligned}
U^\mu \nabla_\mu U^\alpha &= U^\mu \partial_\mu U^\alpha + U^\mu \Gamma^\alpha_{\rho\mu} U^\rho \\
&= \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\partial}{\partial x^\mu} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} + \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \Gamma^\alpha_{\nu\mu} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} \\
&= \ddot{x}^\alpha + \Gamma^\alpha_{\nu\mu} \dot{x}^\nu \dot{x}^\mu = 0
\end{aligned}$$

  with a non-affine parameter: $\nabla_U U = fU$.

- $\nabla_V T = 0$ defines T uniquely along the curve. ODE theory suggests that there is a unique solution for all $T^\mu{}_\nu$.

- Parallel transport T along curves from $p \in \mathcal{M}$ to $q \in \mathcal{M}$ establishes an isomorphism between tensors at p, q. Unlike SR, this is path dependent in GR.

- Parallel transport preserves length of vectors:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \underbrace{(W^\alpha W_\alpha)}_{g_{\alpha\beta} W^\alpha W^\beta} = V^\mu \nabla_\mu (W^\alpha W_\alpha) = 2 W_\alpha \underbrace{V^\mu \nabla_\mu W^\alpha}_{=0} = 0$$

  This means Geodesics do not change their timelike, spacelike or null character.

DEFINITION 6.2. Acceleration along a timelike curve is

$$a^\mu := u^\rho \nabla_\rho u^\mu.$$

The curve is a geodesic if $a^\mu = 0$ or $a^\mu = f u^\mu$.

7 Normal Coordinates

DEFINITION 7.1. Let $\mathcal{M}$ be a manifold, Γ a connection, $p \in \mathcal{M}$. The exponential map

$$e : T_p(\mathcal{M}) \to \mathcal{M}, X_p \mapsto q$$

with q the point a unit affine parameter distance along geodesic through p with tangent X_p.

COMMENT. (1) e is locally bijective;

(2) The vector X_p fixes the parametrisation of the geodesics. One can show that λX_p, $0 \leq \lambda \leq 1$, is mapped to point at affine parameter distance λ along the geodesic of X_p. $(\star)$

DEFINITION 7.2. Let $\{e_\mu\}$ be a basis of $T_p(\mathcal{M})$. The normal coordinates in a neighbourhood of $p \in \mathcal{M}$ are the coordinates that assign to $q \in e(X)$ the coordinates X^μ.

NOTE. The coordinates are not fixed by the vector X. We also need to choose a basis $\{e_\mu\}$.

LEMMA 7.1. *In normal coordinates,* $\Gamma^\mu_{(\nu\rho)} = 0$ *at* p*. If* Γ *is torsion free, then* $\Gamma^\mu_{\nu\rho} = 0$*.*

Proof. From statement $(\star)$, an affinely parametrised geodesic is given by

$$x^\mu(\lambda) = \lambda X^\mu_p.$$

By geodesic equation

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\nu\rho} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda} = 0,$$

we have

$$\Gamma^\mu_{\nu\rho} X^\nu_p X^\rho_p = 0$$

at $p \in \mathcal{M}$, $\forall X \in T_p(\mathcal{M})$. And this suggests $\Gamma^\mu_{(\nu\rho)} = 0$. Together with the condition of free torsion, i.e. $\Gamma^\mu_{[\nu\rho]}$, we have

$$\Gamma^\mu_{\nu\rho} = 0.$$

□

NOTE. In general, $\Gamma^\mu_{\nu\rho} \neq 0$ at $p \neq q$.

LEMMA 7.2. *With Levi-Civita connection, in normal coordinates at $p \in \mathcal{M}$, $\partial_\rho g_{\mu\nu} = 0$.*

Proof. At $p \in \mathcal{M}$, $\Gamma^\rho_{\mu\nu} = 0$. Then

$$2g_{\sigma\rho}\Gamma^\rho_{\mu\nu} = \partial_\nu g_{\sigma\mu} + \partial_\mu g_{\nu\sigma} - \partial_\sigma g_{\mu\nu} = 0$$

Swap σ, μ,

$$\partial_\nu g_{\mu\sigma} + \partial_\sigma g_{\nu\mu} - \partial_\mu g_{\sigma\nu} = 0$$

Add these together, we have

$$\partial_\nu g_{\sigma\mu} = 0.$$

□

COMMENT BY NOTE TAKER. I actually don't like this proof. Why not use $\nabla_\rho g_{\mu\nu} = 0$ together with $\Gamma^\alpha_{\beta\gamma} = 0$?

LEMMA 7.3. *Let $(\mathcal{M}, g)$ be a spacetime with Levi-Civita connection. There exist coordinates at p with $\partial_\rho g_{\mu\nu} = 0$ and*

$$g_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1).$$

Proof. Choose an orthonormal basis $\{e_\mu\}$ for $T_p(\mathcal{M})$. At p,

$$X = X^0 e_0 + X^1 e_1 + X^2 e_2 + X^3 e_3$$

defines normal coordinates $x^\mu = X^\mu$. Argument $(\star)$ suggests that the point with an affine parameter distance λ along geodesic through p with tangent e_0 has coordinate $\lambda(e_0)^\mu = (\lambda, 0, 0, 0)$. In any coordinate system, the tangent vector to the curve $(\lambda, 0, 0, 0)$ is $\partial/\partial x^0 = \partial_0$. Likewise, $\partial_\mu = e_\mu$. Therefore, $\{\partial_\mu\}$ forms an orthonormal basis. Then

$$g_{\mu\nu} = g(\partial_\mu, \partial_\nu) = \eta_{\mu\nu} \quad \text{at } p.$$

□

COMMENT BY NOTE TAKER. I don't quite get why in this case $g(\partial_\mu, \partial_\nu) = \eta_{\mu\nu}$ as the lecturer didn't show it explicitly. I think it is probably from the signature of the spacetime.

DEFINITION 7.3. The local inertial frame at $p \in \mathcal{M}$ is the normal coordinate system with $g_{\mu\nu} = \eta_{\mu\nu}$, $\partial_\sigma g_{\mu\nu} = 0$ at $p \in \mathcal{M}$.

8 The Riemann Tensor

8.1 Commutator

DEFINITION 8.1. The commutator of two vector fields V, W is defined as

$$[V, W]^\alpha = V^\mu \partial_\mu W^\alpha - W^\mu \partial_\mu V^\alpha.$$

It is indeed a vector field since one shows under $x^\alpha \to \tilde{x}^\mu$:

$$\tilde{V}^\nu \tilde{\partial}_\nu \tilde{W}^\mu - \tilde{W}^\nu \tilde{\partial}_\nu \tilde{V}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\gamma}$$

PROPOSITION 8.1.

- $[V,W] = -[W,V]$
- $[V, W+U] = [V,W] + [V,U]$
- $[V, fW] = f[V,W] + V(f)W$
- $[U,[V,W]] + [V,[W,U]] + [W,[U,V]] = 0$ *"Jacobi Identity"*

NOTE. $[\partial_\mu, \partial_\nu] = 0$, i.e. the commutator of coordinate basis is 0.

Conversely, if $V_0, \cdots, V_{m-1},\ m \le \dim(\mathcal{M})$ are vector fields which are linearly independent at $\forall p \in \mathcal{M}$ with all $[V_i, V_j] = 0$, then in a neighbourhood of p, $\exists$ coordinates x^μ such that $V_i = \frac{\partial}{\partial x^i},\ i = 0, 1, \cdots, m-1$.

8.2 Second Derivatives

For a function f, the second derivative is

$$\begin{aligned}\nabla_\nu \nabla_\mu f &= \partial_\nu(\nabla_\mu f) - \Gamma^\rho_{\mu\nu}(\nabla_\rho f) \\ &= \partial_\nu \partial_\mu f - \Gamma^\rho_{\mu\nu}\partial_\rho f \\ &= \nabla_\mu \nabla_\nu f - 2\Gamma^\rho_{[\mu\nu]}\partial_\rho f = \nabla_\mu \nabla_\nu f \quad \text{for torsion free } \Gamma.\end{aligned}$$

For a vector V,

$$\begin{aligned}&\nabla_\alpha \nabla_\beta V^\gamma - \nabla_\beta \nabla_\alpha V^\gamma \\ &= \partial_\alpha \Gamma^\gamma_{\rho\beta} V^\rho + \Gamma^\gamma_{\rho\beta}\partial_\alpha V^\rho + \Gamma^\gamma_{\rho\alpha}\partial_\beta V^\gamma + \Gamma^\gamma_{\rho\alpha}\Gamma^\rho_{\sigma\beta}V^\sigma - (\text{terms with } \alpha \leftrightarrow \beta)\end{aligned}$$

DEFINITION 8.2. The Riemann tensor is defined as

$$R^\gamma{}_{\rho\alpha\beta} := \partial_\alpha \Gamma^\gamma_{\rho\beta} - \partial_\beta \Gamma^\gamma_{\rho\alpha} + \Gamma^\gamma_{\mu\alpha}\Gamma^\mu_{\rho\beta} - \Gamma^\gamma_{\mu\beta}\Gamma^\mu_{\rho\alpha}. \qquad (\dagger)$$

This gives the Ricci identity:

$$R^\gamma{}_{\rho\alpha\beta} V^\rho := \nabla_\alpha \nabla_\beta V^\gamma - \nabla_\beta \nabla_\alpha V^\gamma.$$

Equivalently, we can make the following definition.

DEFINITION 8.3. For three vector fields U, V, W, R is defined by

$$(R(U,V))(W) = \nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U,V]} W. \qquad (\ddagger)$$

One shows for a function f,

$$R(fU,V)W = fR(U,V)W = R(U,fV)W = R(U,V)(fW).$$

The R.H.S. of (‡) is a vector, so it is a linear map on one-forms. So R is a tensor.

To find the components, by

$$[e_\alpha, e_\beta] = 0, \quad \nabla_\alpha e_\beta = \Gamma^\mu_{\beta\alpha} e_\mu$$

we have

$$R(e_\alpha, e_\beta)e_\rho = \nabla_\alpha \nabla_\beta e_\rho - \nabla_\beta \nabla_\alpha e_\rho = \nabla_\alpha(\Gamma^\mu_{\rho\beta} e_\mu) - \nabla_\beta(\Gamma^\mu_{\rho\alpha} e_\mu) = \cdots = R^\nu{}_{\rho\alpha\beta} e_\nu$$

as above (†).

8.3 Symmetries

(1) $R^\alpha{}_{\beta\gamma\delta} = -R^\alpha{}_{\beta\delta\gamma} \quad \Leftrightarrow$

$$\boxed{R^\alpha{}_{\beta(\gamma\delta)} = 0}$$

For torsion-free connection, let (x^μ) be the normal coordinates at $p \in \mathcal{M}$, then:

(2) $\Gamma^\mu_{\nu\rho} = 0$ at p, $\Gamma^\mu_{[\rho\nu]} = 0$ anywhere. We have $R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho}$. Anti-symmetrize on $\nu\rho\sigma$, we have

$$\boxed{R^\mu{}_{[\nu\rho\sigma]} = 0}$$

Since this is tensorial equation, it is valid in all coordinates.

(3) $\nabla_\tau R^\mu{}_{\nu\rho\sigma} = \partial_\tau R^\mu{}_{\nu\rho\sigma} = \partial_\tau \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\tau \partial_\sigma \Gamma^\mu_{\nu\rho}$. Anti-symmetrize on $\rho\sigma\tau$, we have the Bianchi Identities:

$$\boxed{R^\mu{}_{\nu[\rho\sigma;\tau]} = 0}$$

(4) Using Levi-Civita connections, at $p \in \mathcal{M}$, in normal coordinates, $\partial_\mu g_{\nu\rho} = 0$. By $0 = \partial_\mu \delta^\nu{}_\rho = \partial_\mu(g^{\nu\sigma} g_{\sigma\rho}) = g_{\sigma\rho} \partial_\mu g^{\nu\sigma}$, multiplying both sides by $g^{\rho\tau}$, we have $\partial_\mu g^{\nu\tau} = 0$. Then $\partial_\rho \Gamma^\tau_{\nu\sigma} = \frac{1}{2} g^{\tau\mu}(\partial_\rho \partial_\sigma g_{\mu\nu} + \partial_\rho \partial_\nu g_{\sigma\mu} - \partial_\rho \partial_\mu g_{\nu\sigma})$. This gives the components of Riemann tensor with the upstairs index lowered: $R_{\mu\nu\rho\sigma} = \frac{1}{2}(\partial_\rho \partial_\nu g_{\sigma\mu} + \partial_\sigma \partial_\mu g_{\nu\rho} - \partial_\sigma \partial_\nu g_{\rho\mu} - \partial_\rho \partial_\mu g_{\nu\sigma})$. Then we have

$$\boxed{R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}}$$

with (1) we have

$$\boxed{R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta}}$$

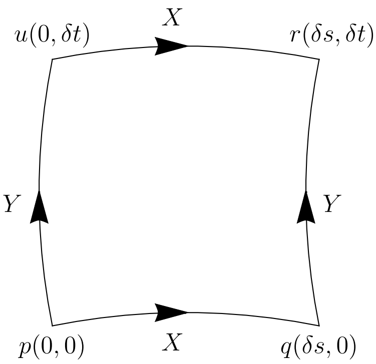
8.4 Parallel Transport and Curvature

Let X, Y be vector fields that are linearly independent everywhere and with $[X, Y] = 0$. Assume torsion-free. We can choose coordinates $(s, t, \cdots)$ such that

$$X = \frac{\partial}{\partial s}, \quad Y = \frac{\partial}{\partial t}.$$

Let $p, q, r, u \in \mathcal{M}$ along the integral curves of X, Y with coordinates

$$p = (0, 0, \cdots), \quad q = (\delta s, 0, \cdots), \quad r = (\delta s, \delta t, \cdots), \quad u = (0, \delta t, \cdots).$$



Let $Z_p \in T_p(\mathcal{M})$ be parallel transported along $p - q - r - u - p$ to get $Z'_p \in T_p(\mathcal{M})$. One can show

$$\boxed{\lim_{\delta s, \delta t \to 0} \frac{(Z'_p - Z_p)^\alpha}{\delta s \delta t} = (R^\alpha{}_{\beta\gamma\delta} Z^\beta Y^\gamma X^\delta)_p}$$

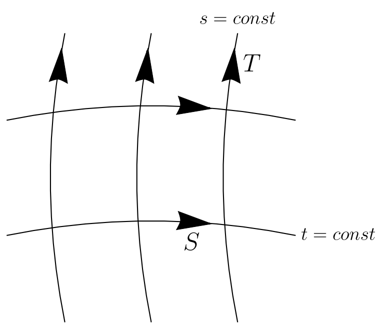
8.5 Geodesics Deviation

Definition 8.4. Let $(\mathcal{M}, \Gamma)$ be a manifold with connection. A one-parameter family of geodesics is a map $\gamma : I \times I' \to \mathcal{M}$ with $I, I' \subset \mathbb{R}$ open and

(i) for fixed s, $\gamma(s,t)$ is a geodesic with affine parameter t;

(ii) locally, $(s,t) \mapsto \gamma(s,t)$ is smooth, bijective and has smooth inverse.

This suggests the family of geodesics forms a two dimensional surface $\Sigma \subset \mathcal{M}$.



Let T be the tangent vector to $\gamma(s = \text{const}, t)$ and S be the tangent vector to $\gamma(s, t = \text{const})$. In coordinates (x^μ), $S^\mu = \mathrm{d}x^\mu / \mathrm{d}s$, then

$$x^\mu(s + \delta s, t) = x^\mu(s,t) + \delta s S^\mu(s,t) + \mathcal{O}\big(\delta s^2\big).$$

Definition 8.5. $\delta s S$ is the deviation vector, it points from one geodesic to a nearby one. The relative velocity of nearby geodesics is defined by $\nabla_T(\delta s S) = \delta s \nabla_T S$. Similarly, the relative acceleration of nearby geodesics is $\delta s \nabla_T \nabla_T S$.

LEMMA 8.2. *If Γ is torsion-free, for vectors fields V, W, we have*

$$\nabla_V W - \nabla_W V = [V, W].$$

Proof.

$$\begin{aligned}
&V^\mu \nabla_\mu W^\alpha - W^\mu \nabla_\mu V^a \\
=&V^\mu \partial_\mu W^\alpha + \Gamma^\alpha_{\rho\mu} V^\mu W^\rho - W^\mu \partial_\mu V^\alpha - \Gamma^\alpha_{\rho\mu} W^\mu V^\rho \\
=&V^\mu \partial_\mu W^\alpha - W^\mu \partial_\mu V^\alpha \\
=&[V, W]^\alpha
\end{aligned}$$

□

THEOREM 8.3. *If Γ is torsion-free, the geodesic deviation is given by*

$$\nabla_T \nabla_T S = R(T, S)T$$

i.e.

$$T^\mu \nabla_\mu (T^\nu \nabla_\nu S^\alpha) = R^\alpha{}_{\lambda\mu\nu} T^\lambda T^\mu S^\nu.$$

Proof. Use coordinates (s, t) on the two dimensional surface Σ spanned by the family of geodesics and extend the coordinates to $(s, t, \cdots)$ in a neighbourhood of Σ. Under this coordinate system

$$S = \frac{\partial}{\partial s}, \quad T = \frac{\partial}{\partial t}$$

from which we easily get $[S, T] = 0$. Since there is no torsion, we have $\nabla_T S - \nabla_S T = [T, S] = 0$. Then

$$\nabla_T \nabla_T S = \nabla_T \nabla_S T = \nabla_S \underbrace{\nabla_T T}_{=0} + R(T, S)T = R(T, S)T$$

□

So $R^\alpha{}_{\beta\gamma\delta}$ measures the geodesic deviation, the manifestation of "curvature".

8.6 The Ricci Tensor

DEFINITION 8.6. The Ricci tensor is defined as

$$\boxed{R_{\alpha\beta} := R^\mu{}_{\alpha\mu\beta}}$$

The Ricci scalar is defined as

$$\boxed{R := g^{\mu\nu} R_{\mu\nu} = R^\mu{}_\mu}$$

And the Einstein tensor is defined as

$$\boxed{G_{\alpha\beta} := R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R}$$

Recall the Bianchi identity, we can show

$$
\begin{aligned}
& R_{\alpha\beta[\gamma\delta;\mu]} = 0 \\
\Rightarrow\quad & \frac{1}{6} g^{\alpha\gamma} g^{\beta\delta} [R_{\alpha\beta\gamma\delta;\mu} + R_{\alpha\beta\delta\mu;\gamma} + R_{\alpha\beta\mu\gamma;\delta} - \underbrace{R_{\alpha\beta\delta\gamma;\mu}}_{=-R_{\alpha\beta\gamma\delta;\mu}} - R_{\alpha\beta\mu\delta;\gamma} - R_{\alpha\beta\gamma\mu;\delta}] = 0 \\
\Rightarrow\quad & \frac{1}{3} g^{\alpha\gamma} g^{\beta\delta} (R_{\alpha\beta\gamma\delta;\mu} + R_{\alpha\beta\delta\mu;\gamma} + R_{\alpha\beta\mu\gamma;\delta}) = 0 \\
\Rightarrow\quad & R_{;\mu} - g^{\alpha\gamma} R_{\alpha\mu;\gamma} - g^{\beta\delta} R_{\beta\mu;\delta} = 0 \\
\Rightarrow\quad & \nabla_\mu R - 2\nabla_\gamma R^\gamma{}_\mu = -2\nabla^\gamma (R_{\gamma\mu} - \frac{1}{2} g_{\gamma\mu} R) = 0 \\
\Rightarrow\quad & \boxed{\nabla^\mu G_{\mu\alpha} = 0} \quad \text{This is the contracted Bianchi identity.}
\end{aligned}
$$

References

[1] U. Sperhake *Part II General Relativity Short Lecture Notes*

[2] U. Sperhake *Part II General Relativity Long Lecture Notes*